ON THE PROBLEM OF DIFFRACTION OF SOUND FROM A CIRCULAR APERTURE

(K ZADACHE O DIFRAKTSII ZVUKA OT KRUGLOGO OTVERSTIIA)

PMM Vol.24, No.5, 1960, pp. 931-932

N.A. ROSTOVTSEV
(Komsomol'sk-na-Armure)

(Received 22 June 1960)

The problem of the title is considered here for the particular case of axial symmetry. In contrast with known methods of solution based on the application of expansions in terms of spherical wave functions (see the bibliography of [1,2]), a different approach is given here, which consists of the representation of the diffraction potential by means of a single integral. The boundary conditions lead to a Fredholm integral equation of the second kind for the auxiliary function which appears under the integral sign. Even though this equation is not solved in final form, one can obtain, nevertheless, without difficulty an approximate solution by the method of a small parameter, which in this problem is a dimensionless quantity ka (where $k = 2\pi/\lambda$ is the wave number, a is the radius of the aperture). This approach is a modification of the method of the complex potential which was developed by the author for a restricted class of problems on the Newtonian potential for a half-space [3].

1. Let us introduce cylindrical coordinates r, θ, z. For the plane $z = 0$ we select the plane of a rigid screen in which there is an aperture given by $r \leqslant a$. The oscillation source is assumed to lie in the half-space $z < 0$. The wave-potential $\phi(r, z)$ which corresponds to the given source distribution in the absence of the aperture is assumed to be known. The phase factor is taken in the form $e^{-j\omega t}$. Therefore, the solutions will decrease at infinity in both halves of the plane as $R^{-1}e^{ikR}$, where $R = \sqrt{(r^2 + z^2)}$ (conditions of radiation). Let us denote by $\psi(r, z)$ and $\chi(r, z)$ the diffraction potentials in the half-spaces $z < 0$ and $z > 0$, respectively. The continuity of the velocity over the aperture is guaranteed if one sets $\psi(r, z) = -\chi(r, -z)$. The remaining boundary conditions will be

$$2[\chi] = \varphi(r, 0) = 2f(r) \ \text{at the aperture}, \qquad \left[\frac{\partial \chi}{\partial z}\right] = 0 \ \text{outside the aperture} \tag{1.1}$$

We shall try to find a solution by means of the superposition of elementary solutions of the form

$$[r^2 + (z + iv)^2]^{-1/2} \exp\{ik[r^2 + (z + iv)^2]^{-1/2}\}$$

which have singularities at the pure imaginary points of the z-axis, $z = 0$ and $z = -iv$. The square root of a complex number is taken under the condition that the real part be positive. Therefore, if $w = u + iv$ is a complex number, then $(r^2 + w)^{1/2}$ is evaluated in the plane cut along the imaginary axis from $+ir$ to $+i\infty$ and from $-ir$ to $-i\infty$. The required particular solution will be taken in the form

$$\theta(r, z; s) = \frac{-1}{2} \int_{-s}^{+s} \frac{\exp[ik\sqrt{r^2 + (z + iv)^2}]}{\sqrt{r^2 + (r + iv)^2}}\, dv = \frac{i}{2} \int_{z-is}^{z+is} \frac{\exp[ik\sqrt{r^2 + w^2}]}{\sqrt{r^2 + w^2}}\, dw \tag{1.2}$$

It represents the potential due to sources that are uniformly distributed in the region $z = -iv$, $-s < v < s$. We have $\theta(r, z; 0) = 0$, and the following formulas for the derivatives:

$$\frac{\partial \theta}{\partial z} = \frac{i}{2} \left\{ \frac{\exp[ik\sqrt{r^2 + (z + is)^2}]}{\sqrt{r^2 + (z + is)^2}} - \frac{\exp[ik\sqrt{r^2 + (z - is)^2}]}{\sqrt{r^2 + (z - is)^2}} \right\} \tag{1.3}$$

$$\frac{\partial \theta}{\partial s} = \frac{-1}{2} \left\{ \frac{\exp[ik\sqrt{r^2 + (z + is)^2}]}{\sqrt{r^2 + (z + is)^2}} + \frac{\exp[ik\sqrt{r^2 + (z - is)^2}]}{\sqrt{r^2 + (z - is)^2}} \right\} \tag{1.4}$$

Next, let us introduce a complex function $g(s)$, of the real variables, $0 < s < \infty$, whose real and imaginary parts are piece-wise monotone, and which is such that $g(\infty) = 0$.

We construct the diffraction potential in the region $z > 0$ in the form of a Stieltjes integral

$$\chi(r, z) = \int_0^\infty \theta(r, z; s)\, dg(s) \tag{1.5}$$

Let us assume that the integral converges and that it is permissible to differentiate with respect to the parameters. This will be justified in the process of the actual construction of the solution. Integrating by parts, we reduce $\chi(r, z)$ to the form

$$\chi(r, z) = \frac{1}{2} \int_0^\infty \left\{ \frac{\exp[ik\sqrt{r^2 + (r + is)^2}]}{\sqrt{r^2 + (z + is)^2}} + \frac{\exp[ik\sqrt{r^2 + (z - is)^2}]}{\sqrt{r^2 + (z - is)^2}} \right\} g(s)\, ds \tag{1.6}$$

Differentiating (1.5), we obtain

$$\frac{\partial\chi}{\partial r} = \frac{i}{2}\int_0^\infty \left\{\frac{\exp\left[ik\sqrt{r^2+(z-is)^2}\right]}{\sqrt{r^2+(z+is)^2}} - \frac{\exp\left[ik\sqrt{r^2+(z+is)^2}\right]}{\sqrt{r^2+(z-is)^2}}\right\} dg\,(s) \tag{1.7}$$

Assuming that $z = +0$, we obtain the boundary values

$$[\chi] = \int_0^r \frac{\exp\left[ik\sqrt{r^2-s^2}\right]}{\sqrt{r^2-s^2}}\, g\,(s)\, ds + i\int_r^\infty \frac{\sinh k\sqrt{s^2-r^2}}{\sqrt{s^2-r^2}}\, g\,(s)\, ds \tag{1.8}$$

$$\left[\frac{\partial\chi}{\partial z}\right] = \int_r^\infty \frac{\cosh k\sqrt{s^2-r^2}}{\sqrt{s^2-r^2}}\, dg\,(s) \tag{1.9}$$

It is clear that if we impose the condition that $g(s) = 0$ for all $s > a$ then the second boundary condition will be satisfied, and the previously made assumptions on the convergence and differentiability with respect to the parameters of the integral will thereby be justified. Furthermore, since the integration is in reality carried out over a finite interval $0 \leqslant s \leqslant a$, one can see that the solution (1.6) satisfies the condition of radiation. The first boundary condition leads, however, to the integral equation

$$\int_0^r \frac{\cos k\sqrt{r^2-s^2}}{\sqrt{r^2-s^2}}\, g\,(s)\, ds + i\int_0^a \frac{\sin k\sqrt{r^2-s^2}}{\sqrt{r^2-s^2}}\, g\,(s)\, ds = f\,(r) \tag{1.10}$$

This equation can be reduced to a Fredholm equation of the second kind if one multiplies each of its sides by $[r \cosh k\sqrt{(u^2 - r^2)}]\,/\sqrt{(u^2 - r^2)}$ integrates with respect to r from $r = 0$ to $r = u$, and then differentiates with respect to u. One thus obtains

$$g\,(u) + \frac{2i}{\pi}\int_0^a K(u,\, s)\, g\,(s)\, ds = \frac{2}{\pi}\int_0^a \frac{r\cosh k\sqrt{u^2-r^2}}{\sqrt{u^2-r^2}}\, f\,(r)\, dr \tag{1.11}$$

where

$$K\,(u,\, s) = \frac{1}{2}\left\{\frac{\sinh k\,(u+s)}{u+s} + \frac{\sinh k\,(u-s)}{u-s}\right\} \tag{1.12}$$

The original form (1.10) is, however, more convenient than (1.11) for practical computations of the first few terms of the expansion in powers of the small parameter k (or ka).

2. In case of a planar wave, we have

$$\varphi\,(r,\, z) = e^{ikr} + e^{-ikz} = 2\cos kz, \qquad f\,(r) = \tfrac{1}{2}\varphi\,(r,\, 0) = 1$$

The substitution of

$$g\,(s) = g_0\,(s) + kg_1\,(s) + k^2 g_2\,(s) + \ldots \tag{2.1}$$

into (1.9) leads to a recurring system in which $g_{n+1}(s)$ is found from the preceding $g_k(s)$, $(k = 0, 1, \ldots, n)$ as a solution of Abel's equation with the kernel $(r^2 - s^2)^{-1/2}$ and a polynomial right-hand side. The evaluations are elementary. The first terms of the expansion are

$$g(s) = \frac{2}{\pi} - i\frac{4}{\pi^2}k + \left(\frac{2}{\pi}s^2 - \frac{16}{\pi^3}\right)\frac{k^2}{2!} - i\left\{\frac{4}{\pi}s^2 - \frac{16}{3\pi^2}\left(\frac{18}{\pi^2} - 1\right)\right\}\frac{k^3}{3!} + \\ + \left\{\frac{2}{\pi}s^4 - \frac{32}{\pi^2}s^2 + \frac{64}{3\pi^3}\left(\frac{12}{\pi^2} - 1\right)\right\}\frac{k^4}{4!} + \ldots \tag{2.2}$$

It is assumed here for the sake of simplicity that $a = 1$. In case of relatively large values of ka, from 0.5 up, it is more convenient to use Equation (1.11). Its kernel (1.12) can be approximated quite well by means of polynomials. For numerical analysis it is best to use the expansion of sinh kx in terms of Chebychev polynomials $T_n(x)$.

BIBLIOGRAPHY

1. Strutt, M., *Funktsii Liame, Mat'e i rodstvennye im v fizike i tekhnike (Lamé, Matthieu and Related Functions in Physics and Technology)*. (Russian translation of the German book: Strutt, M.J.O., *Lamesche, Mathieusche und verwandte Funktionen in Physik und Technik,* 1932). ONTI, 1935.

2. Skuchik, E., *Osnovy akustiki (Fundamentals of Acoustics)*, Vol. 1. IIL, 1958.

3. Rostovtsev, N.A., Kompleksnye potentsialy v zadache o shtampe, kruglom v plane (Complex potentials in the problem of a stamp with circular base). *PMM* Vol. 21, No. 1, 1957.

Translated by H.P.T.